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Properly Efficient and Efficient Solutions for Vector Maximization Problems in Euclidean Space

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Submitted by George Leitmann

Recently Benson proposed a definition for extending Geoffrion's concept of proper efficiency to the vector maximization problem in which the domination cone K is any nontrivial, closed convex cone. We give an equivalent definition of his notion of proper efficiency. Our definition, by means of perturbation of the cone K, seems to offer another justification of Benson's choice above Borwein's extension of Geoffrion's concept. Our result enables one to prove some other theorems concerning properly efficient and efficient points. Among these is a connectedness result.

INTRODUCTION

Let $Y \subseteq R^n$ be a nonempty set containing the origin. Let K be a closed convex cone in R^n such that $K \neq \{0\}$ and $K \neq R^n$. Then $y = 0$ is said to be *efficient* iff $Y \cap K = \{0\}$. The point $y = 0$ is *properly efficient* in the sense of *Geoffrion* [1] iff

(1) $y = 0$ is efficient with respect to $K = R^n_+$.

(2) There exists a scalar $M > 0$ such that for each $i \in \{1, 2, ..., n\}$ and each $y = (y_i)_{i=1}^n \in Y$ satisfying $y_i > 0$, there exists at least one $j \in \{1, 2, ..., n\}$ with $y_i \leqslant -My_j$.

In order to define Borwein's and Benson's extension to the case where K is any nontrivial closed convex cone we shall state, for the sake of completeness, the definitions of *tangent cone* and *projecting cone*.

Let $C \subseteq R^n$ and $\omega \in C$. The *tangent cone* to C at ω, denoted by $T(C, \omega)$, is the set of all limits of the form

$$h = \lim \lambda_i(\omega^i - \omega), \qquad \text{where } \{\lambda_i\} \subseteq R_+, \text{ and } \{\omega^i\} \subseteq C \text{ with } \omega_i \to \omega.$$

The *projecting cone* of C, denoted $P(C)$, is the set of all points of the form

$$h = \lambda\bar{\omega}, \qquad \text{where } \lambda \in R_+ \text{ and } \bar{\omega} \in C.$$

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Now we can state the promised extensions of proper efficiency. The point $y = 0$ is *properly efficient* in the sense of *Borwein* [2] iff

(1) $y = 0$ is efficient.

(2) $T(Y - K, 0) \cap K = \{0\}$.

The point $y = 0$ is *properly efficient* in the sense of *Benson* [3] iff

(1) $y = 0$ is efficient.

(2) $\mathrm{cl}\, P(Y - K) \cap K = \{0\}$.

As Benson proved in [3], when $K = R^n_+$ then his notion of proper efficiency is equivalent to proper efficiency in the sense of Geoffrion. Further he showed that this is not the case for Borwein's notion of proper efficiency.

In the sequel we will prove that $y = 0$ is properly efficient in the sense of Benson if and only if $y = 0$ is efficient with respect to a small disturbance of the cone K, by which we give another justification of Benson's choice. Further we will prove by means of our characterization of properly efficient points in the sense of Benson some other results, including a connectedness result concerning these properly efficient points.

Approximating Cones

Let $\{K_\epsilon, 0 < \varepsilon \leqslant \bar{\varepsilon}\}$ be a collection of closed convex cones in R^n.

Definition. The family $\{K_\epsilon\}$ is said to be *K-approximating* iff

(1) $K \backslash \{0\} \subseteq \mathrm{int}\, K_\epsilon, \qquad \forall \varepsilon \in (0, \bar{\varepsilon}]$.

(2) $K_\epsilon \subseteq K_{\epsilon'}$, *whenever* $\varepsilon \leqslant \varepsilon'$.

(3) *Let* $\{k_\epsilon\}$ *be a sequence of points with* $\varepsilon \downarrow 0$, $k_\epsilon \in K_\epsilon$, *and* $|k_\epsilon| = 1$; *then any convergent subsequence of such a sequence* $\{k_\epsilon\}$ *has its limit in K. For brevity this will be written as* $\lim\{k_\epsilon, \varepsilon \downarrow 0, k_\epsilon \in K_\epsilon, |k_\epsilon| = 1\} \in K$.

First of all we will answer the question when K has a family $\{K_\epsilon\}$ of approximating cones.

Theorem 1. *The cone K has a family of approximating cones* $\{\tilde{K}_\epsilon\}$ *if and only if* $K \cap (-K) = \{0\}$.

Proof. Suppose that $K \cap (-K) \neq \{0\}$ and $\{K_\epsilon\}$ is K-approximating; then $0 \in \mathrm{int}\, K_\epsilon$, hence $K_\epsilon = R^n$ and we arrive at a contradiction with the third aspect of the definition above.

Now suppose that $K \cap (-K) = \{0\}$. We will construct a family $\{\tilde{K}_\epsilon\}$ of K-approximating cones as follows.

Defining $K^* = \{k^* \in R^n \mid k^*k \geqslant 0, \quad \forall k \in K\}$ it is well-known that $K \cap (-K) = \{0\}$ implies that $K^* - K^* = R^n$, hence that $\operatorname{int} K^* \neq \emptyset$. Take $k^* \in \operatorname{int} K^*$, then $k^*k > 0$, $\forall k \in K\backslash\{0\}$, for suppose that $k^*k = 0$, then $(k^* + u^*)k \geqslant 0$, for all $u^* \in U^*$ a neighborhood of the origin in R^n, hence $u^*k \geqslant 0$. $\forall u^* \in U^*$, and therefore $k = 0$. Now take an arbitrary $\delta > 0$ and define

$$K(\delta) = \{k \in K \mid k^*k = \delta\}.$$

Further define $U(\varepsilon) = \{x \in R^n \mid |x| \leqslant \varepsilon\}$ and

$$\tilde{K}_\epsilon = \bigcup_{\lambda \geqslant 0} \lambda(K(\delta) + U(\varepsilon)).$$

Almost immediately it follows that

(a) $\tilde{K}_\epsilon$ is a convex cone with nonempty interior.

(b) $K\backslash\{0\} \subseteq \operatorname{int} \tilde{K}_\epsilon$.

In order to complete the proof we will show that $K(\delta)$ is compact. As the closedness of $K(\delta)$ is obvious there remains the proof of its boundedness. Suppose, to the contrary, that there is a sequence $\{k^i \mid |k^i| \to +\infty\} \subseteq K(\delta)$; then

$$k^* \frac{k^i}{|k^i|} = \frac{\delta}{|k^i|} \to 0,$$

but

$$\frac{k^i}{|k^i|} \to \bar{k} \in K \qquad \text{with} \quad \bar{k} \neq 0,$$

and we arrive at a contradiction.

Now we will prove that

(c) $\tilde{K}_\epsilon$ is closed for ε small enough.

Take a sequence of points $\{k^i\} \subseteq \tilde{K}_\epsilon$; then

$$k^i = \lambda_i(k(\delta)^i + u(\varepsilon)^i), \qquad \lambda_i \geqslant 0,\ k(\delta)^i \in K(\delta),\ u(\varepsilon)^i \in U(\varepsilon).$$

Suppose $k^i \to \bar{k}$ and, without loss of generality, $k(\delta)^i \to k(\delta) \in K(\delta)$, and $u(\varepsilon)^i \to u(\varepsilon) \in U(\varepsilon)$. For ε small enough we have that

$$k^*(\tilde{k}(\delta) + \tilde{u}(\varepsilon)) > 0, \qquad \forall \tilde{k}(\delta) \in K(\delta),\ \forall \tilde{u}(\varepsilon) \in U(\varepsilon).$$

But $\lambda_i \to +\infty$ implies that $k(\delta) + u(\varepsilon) = 0$, a contradiction for ε small enough. Hence we may assume that $\lambda_i \to \bar{\lambda} \geqslant 0$ and therefore

$\bar{k} = \bar{\lambda}(k(\delta) + u(\varepsilon))$, by which we proved that $\tilde{K}_\varepsilon$ is closed for ε small enough.

It remains to be proved that

(d) $\lim\{k_\varepsilon, \varepsilon \downarrow 0, k_\varepsilon \in \tilde{K}_\varepsilon, |k_\varepsilon| = 1\} \in K$.

Let $k_\varepsilon = \lambda_\varepsilon(k(\delta)_\varepsilon + u_\varepsilon)$ with $k(\delta)_\varepsilon \in K(\delta)$ and $u_\varepsilon \in U(\varepsilon)$, and $\lambda_\varepsilon \geqslant 0$. Further let $\lim k_\varepsilon = \bar{k}$ and $\lim k(\delta)_\varepsilon = k(\delta)$. Now suppose that $\lambda_\varepsilon \to +\infty$; hence $k(\delta) = 0$, a contradiction. Hence we may suppose that $\lambda_\varepsilon \to \bar{\lambda} \geqslant 0$ and therefore $\bar{k} = \bar{\lambda} k(\delta) \in K$, and the proof is complete.

An Alternative Characterization of Properly Efficient Points in the Sense of Benson

Theorem 2. *Let $0 \in Y \subseteq R^n$, $Y \neq \{0\}$, and let $K \subseteq R^n$ be a closed convex cone such that $K \neq \{0\}$ and $K \neq R^n$. Then in order for 0 to be properly efficient in the sense of Benson it is necessary that K admit for a family $\{K_\varepsilon\}$ of K-approximating cones.*

Further, when $\{K_\varepsilon\}$ is a family of K-approximating cones, we have that $0 \in Y$ is properly efficient if and only if there is a cone K_ε such that 0 is efficient with respect to K_ε.

Proof. Suppose that K does not admit for a family of K-approximating cones; then, by Theorem 1, $K \cap (-K) \neq \{0\}$, hence, as $0 \in Y$,

$$P(Y - K) \cap K \neq \{0\}.$$

Let $\{K_\varepsilon\}$ be a family of K-approximating cones. Suppose that 0 is efficient with respect to K_ε, in other workds, $K_\varepsilon \cap Y = \{0\}$. Suppose to the contrary that there is a $\bar{k} \in K\backslash\{0\}$ such that

$$\bar{k} \in \text{cl } P(Y - K),$$

that is,

$$\bar{k} = \lim k^i, \qquad k^i = \lambda_i(y^i - \bar{k}^i), \qquad \lambda^i > 0, y^i \in Y, \bar{k}^i \in K.$$

As $\bar{k} \neq 0$ we may assume that $k^i \in K\backslash\{0\} \subseteq \text{int } K_\varepsilon$. We now have

$$y_i = \frac{k^i + \lambda_i \bar{k}_i}{\lambda_i} \in \text{int } K_\varepsilon,$$

a contradiction; hence $\text{cl } P(Y - K) \cap K = \{0\}$.

Now suppose that $Y \cap K_\varepsilon \neq \{0\}$ for all cones K_ε; hence there is a sequence $\{k_\varepsilon \mid k_\varepsilon \in Y \cap K_\varepsilon, \ k_\varepsilon \neq 0\}$. Now define $k'_\varepsilon = k_\varepsilon / |k_\varepsilon|$; then $k'_\varepsilon \to \bar{k} \in K$ with $|\bar{k}| = 1$. Further $k'_\varepsilon = (1/|k_\varepsilon|)(k_\varepsilon - 0) \in P(Y - K)$; hence

$\bar{k} \in \operatorname{cl} P(Y-K)$ and therefore $\operatorname{cl} P(Y-K) \cap K \neq \{0\}$ and we are done. Denoting the set of properly efficient points of Y with respect to K by Y_{PRE} and the set of efficient points of Y with respect to K_ϵ by Y_ϵ we therefore have:

COROLLARY 1. $Y_{\mathrm{PRE}} = \bigcup_\epsilon Y_\epsilon$.

Denoting the set of points y of Y with the property that

$$(y + \operatorname{int} K_\epsilon) \cap Y = \{\emptyset\} \text{ by } Y_{W\epsilon},$$

it is easy to see by inspection of the proof of Theorem 2 that the following does hold.

COROLLARY 2. $Y_{\mathrm{PRE}} = \bigcup_\epsilon Y_{W\epsilon}$.

The interpretation of Theorem 2 is clear: a point is properly efficient in the sense of Benson if and only if it remains efficient under a small perturbation of the cone K. By means of the device of K-approximating cones developed above, we will prove some other theorems.

THEOREM 3. *Let Y be such that $Y-K$ is convex. Further, let*

$$B(k^*) = \{\bar{y} \in Y \mid k^*\bar{y} = \max_{y \in Y} k^*y\};$$

then

$$Y_{\mathrm{PRE}} = \bigcup_{k^* \in \operatorname{int} K^*} B(k^*).$$

Proof. Take an arbitrary family $\{K_\epsilon\}$ of K-approximating cones. Let $\bar{y} \in Y_{\mathrm{PRE}}$; then $(\bar{y} + \operatorname{int} K_\epsilon) \cap Y = \emptyset$, for some cone K_ϵ. Applying Hahn and Banach's separation theorem leads to the existence of a $k_\epsilon^* \in K_\epsilon^*$, $k_\epsilon^* \neq 0$ such that $k_\epsilon^*(\bar{y} + k_\epsilon) \geqslant k_\epsilon^* y$, $\forall k_\epsilon \in K_\epsilon$, $\forall y \in Y$. But $K \backslash \{0\} \subseteq \operatorname{int} K_\epsilon$; hence $k_\epsilon^* \in K^*$, even $k_\epsilon^* \in \operatorname{int} K^*$, as $k_\epsilon^* k_\epsilon > 0$, $\forall k_\epsilon \in \operatorname{int} K_\epsilon$, and also $\bar{y} \in B(k_\epsilon^*)$. Now take an arbitrary $k^* \in \operatorname{int} K^*$ and an $\bar{y} \in B(k^*)$. Take, as in the proof of Theorem 1, a $\delta > 0$, define $K(\delta)$ and $\tilde{K}_\epsilon$ as in that theorem. Take $\varepsilon > 0$ such that $k^*(y(\delta) + u(\varepsilon)) > 0$, $\forall y(\delta) \in K(\delta)$, $\forall u(\varepsilon) \in U(\varepsilon)$. Then we conjecture that

$$(\bar{y} + \operatorname{int} \tilde{K}_\epsilon) \cap Y = \emptyset.$$

This can be seen as follows. By construction of $\tilde{K}_\epsilon$ we have that $k^* k_\epsilon \geqslant 0$, $\forall k_\epsilon \in \tilde{K}_\epsilon$, further $k^* \neq 0$. Now suppose there is a point $k_\epsilon \in \operatorname{int} \tilde{K}_\epsilon$ such that $\bar{y} + k_\epsilon \in Y$, then $k^*(y + k_\epsilon) > k^*\bar{y}$, a contradiction. Together with Corollary 2 this implies that $\bar{y} \in Y_{\mathrm{PRE}}$, and we are done.

We would like to remark that Theorem 3 actually is Theorem 4.2 from [3] and Theorem 2 in [2]. Our method of proof, however, is different.

The following result generalizes Theorem 4.3 from [4].

THEOREM 4. *Let $Y \subseteq R^n$ be such that $0 \in Y$. If* $\sup\{|y| \mid y \in Y \cap K\} = +\infty$, *then* $Y_{\mathrm{PRE}} = \varnothing$.

Proof. Suppose $y^* \in Y_{\mathrm{PRE}}$, then for some $\tilde{K}_\epsilon$, where $\tilde{K}_\epsilon$ is as in Theorem 1, we have that $(y^* + \tilde{K}_\epsilon) \cap Y = \{y^*\}$. Take an arbitrary $k \in K \backslash \{0\}$; then, as $K \backslash \{0\} \subseteq \operatorname{int} \tilde{K}_\epsilon$, $\lambda k \in y^* + \tilde{K}_\epsilon$ for λ sufficiently large. Define $\lambda(k) = \inf\{\lambda \mid \lambda k \in y^* + \tilde{K}_\epsilon\}$ and $\bar{\lambda} = \sup\{\lambda(k) \mid k \in K(\delta)\}$, where $K(\delta)$ is as in Theorem 1. Suppose to the contrary that there is a sequence $\lambda(k^i) \to +\infty$ for some sequence $\{k^i\} \subseteq K(\delta)$; then

$$(\lambda(k^i) - \bar{\delta})\, k^i - y^* \notin \tilde{K}_\epsilon$$

for some $\bar{\delta} > 0$. Hence

$$k^i - \frac{y^*}{\lambda(k^i) - \bar{\delta}} \notin \tilde{K}_\epsilon,$$

but this contradicts the construction of $\tilde{K}_\epsilon$; hence $\bar{\lambda}$ is finite, hence $(\bar{\lambda} + \delta)k \in y^* + \tilde{K}_\epsilon$ for all $k \in K(\delta)$, and all $\delta > 0$. As $K(\delta)$ is compact, (see Theorem 1) it follows that $\sup\{|k| \mid k \in K(\delta)\} < \infty$; hence there is a number $\bar{\delta} > 0$ such that $y \in K$ and $|y| \geqslant \bar{\delta}$ implies that $y \in y^* + \tilde{K}_\epsilon$. Together with $(y^* + \tilde{K}_\epsilon) \cap Y = \{y^*\}$ this implies that $\sup\{|y| \mid y \in Y \cap K\} < +\infty$, and we are done.

We would like to remark that Theorem 4 does hold for every norm $|\cdot|$ on R^n, because all norms on R^n are equivalent. Notice further that the main argument in the theorem above is easy and geometrical: For any $y \in K$, there is a $\lambda > 0$ such that λy is in $y^* + \tilde{K}_\epsilon$, regardless of the position of y^*.

The next result generalizes Corollary 3.2 from [4] and is implied by Theorem 4.1 from [5].

THEOREM 5. *Let $Y \subseteq R^n$ be convex and $Y - K$ closed. If* $\sup\{|y| \mid y \in Y \cap K\} = +\infty$, *then Y has no efficient points.*

Proof. The efficient points of Y are exactly those of $Y - K$. We denote these points by Y_E. Adapting a result of Arrow *et al.* [6 (see also [5, Theorem 5.5]). and applying Theorem 3 we have that

$$(Y - K)_{\mathrm{PRE}} \subseteq Y_E \subseteq \overline{(Y - K)_{\mathrm{PRE}}}.$$

Further, $\sup\{|y| \mid y \in Y \cap K\} = \sup\{|y| \mid y \in (Y - K) \cap K\} = +\infty$, as $Y \cap K \subseteq (Y - K) \cap K$; hence, applying Theorem 4, it follows that $(Y - K)_{\mathrm{PRE}} = \varnothing$, and therefore $Y_E = \varnothing$, and we are done.

A Connectedness Result

The following result is due to Naccache [7].

LEMMA 1. *Let $Y \subseteq R^n$ be nonempty, compact and convex. Let $\tilde{K} \subseteq R^n$ be a closed convex cone with nonempty interior. Then the set of efficient points of Y with respect to $\tilde{K}$ is a connected set.*

THEOREM 6. *Let $Y \subseteq R^n$ be nonempty, compact and convex. Let $K \subseteq R^n$ be a closed convex cone such that $K \neq R^n$, $K \neq \{0\}$. Then Y_{PRE} is connected.*

Proof. From Corollary 1 to Theorem 2 we know that $Y_{\mathrm{PRE}} = \bigcup_{0<\epsilon\leqslant\bar{\epsilon}} Y_\epsilon$. Without loss of generality we may take $\bar{\epsilon}$ such that $Y_{\bar{\epsilon}} \neq \varnothing$; hence by the definition of K-approximating cones it follows that $\bigcap Y_\epsilon = Y_{\bar{\epsilon}} \neq \varnothing$. Now it follows from Lemma 1 and a result of Whyburn and Duda [8] that Y_{PRE} is connected, and we are done.

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